UDC 62-50

ENCOUNTER-EVASION PROBLEMS IN SYSTEMS WITH A SMALL PARAMETER IN THE DERIVATIVES

PMM Vol. 38, № 5, 1974, pp. 771-779

N. N. KRASOVSKII and V. M. RESHETOV

(Sverdlovsk)

(Received February 18, 1974)

We examine encounter evasion game problems for a linear controlled system described by differential equations with a small parameter in a part of the derivatives [1]. On the basis of the procedure of control with a leader [2, 3] we construct a strategy which ensures an encounter or an evasion generated by its motions relative to a specified closed target set within the limits of another closed set of phase coordinates. In particular, we examine the problem of evasion during an arbitrarily large time interval. The work relies on the formalization of differential games given in [2]. As an example we consider the evasion problem for a system asymptotic with respect to the small parameter to a system described in [4].

1. Let a controlled system be described by the vector differential equation

$$x^{\cdot} = Ax + Bu + Cv \tag{1.1}$$

Here x is an n-dimensional phase coordinate vector; u and v are $r^{(1)}$- and $r^{(2)}$-dimensional control vectors of the first and second players; A, B, C are constant matrices of the appropriate dimensions; the first and second players' controls are subject to the conditions

$$u \in P, \qquad v \in Q^{[\alpha]} \tag{1.2}$$

where P and Q are bounded convex sets in vector spaces $\{u\}$ and $\{v\}$. The symbol R^α denotes the Euclidean α-neighborhood of set R and the symbol $R^{[\alpha]}$ denotes the closed Euclidean α-neighborhood of R. The vectors being considered are treated as column-vectors. We use the terms strategies, motions, Euler polygonal lines, and the notation corresponding to them in the same sense as they were defined in [2].

Suppose that certain closed sets M and N are specified in a k-dimensional subspace $\{x\}_k$ of the n-dimensional phase space $\{x\}$. We solve the following problem facing the second player. This problem is to construct a strategy V operating in a leader-control plan and ensuring for all motions, namely, the Euler polygonal lines $x_\Delta[t]$ generated by this strategy, the contact

$$\{x_\Delta[\tau^*]\}_k \in M^{[\varepsilon]}, \quad \{x_\Delta[t]\}_k \in N^{[\varepsilon]} \qquad (t_0 \leqslant t \leqslant \tau^*) \tag{1.3}$$

with the neighborhood $M^{[\varepsilon]}$ within the limits of the neighborhood $N^{[\varepsilon]}$ under any action taken by the first player compatible with constraints (1.2). The value τ^* in (1.3) can depend upon the motion. In particular, set N can coincide with the whole space $\{x\}_k$, in which case the second condition in (1.3) is automatically fulfilled. Set M can, in general, be absent and then the first condition in (1.3) is eliminated, while the second one must be fulfilled for $t_0 \leqslant t \leqslant \vartheta$, where the number ϑ is stipulated by the conditions of the problem. In particular, when $\vartheta = \infty$ we obtain one version of the evasion problem on an infinite time interval. We note further that the additional condition $\tau^* \leqslant \vartheta$ can be added to (1.3) in the general case.

Assume that system (1.1) can be represented as

$$z^{\cdot} = A_1 z + D_1 y + B_1 u + C_1 v, \quad \mu y^{\cdot} = A_2 z + D_2 y + C_2 v \qquad (1.4)$$

Here z is a k-dimensional vector, where the space $\{z\}_k$ coincides exactly with the subspace $\{x\}_k$; y is an $(n-k)$-dimensional vector; $\mu > 0$ is a small parameter. The problem is then stated as follows. For a specified initial position $\{t_0, z_0, y_0\}$ and $\varepsilon > 0$ we are required to find a leader-control procedure V which, for sufficiently small values of parameter μ and for a sufficiently small partitioning step $\delta = \sup_i (\tau_{i+1} - \tau_i)$ $(i = 0, 1, \ldots)$ on the t-axis, ensures the contact

$$\{z_\Delta [\tau^*]\} \in M^{[\varepsilon]}, \quad \{z_\Delta [t]\} \in N^{[\varepsilon]} \quad (t_0 \leqslant t \leqslant \tau^*) \qquad (1.5)$$

for all approximating Euler polygonal lines $x_\Delta [t] = \{z_\Delta [t] = z_\Delta [t, t_0, z_0, y_0, V, u[\cdot]], \; y_\Delta [t] = y_\Delta [t, t_0, z_0, y_0, V, u[\cdot]]\}$.

We approach this problem in the following way. As in [1] we set $\mu = 0$ in (1.4); assuming that the denominator $| D_2^{-1} | \neq 0$, from the second equation we find the quantity

$$y^\circ (z, v) = -D_2^{-1} A_2 z - D_2^{-1} C_2 v \qquad (1.6)$$

We set up an auxilliary differential equation

$$z^{\cdot\circ} = A_1 z^\circ + D_1 y^\circ (z^\circ, v) + B_1 u + C_1 v = A z^\circ + B u + C v \qquad (1.7)$$

$$(A = A_1 - D_1 D_2^{-1} A_2, \quad B = B_1, \quad C = C_1 - D_1 D_2^{-1} C_2)$$

Following [2, 3] we compare Eq. (1.7) with the equation of motion of the leader

$$w^{\cdot} = A w + B u_* + C v_* \qquad (1.8)$$

for controls u_* and v_* subject to the conditions

$$u_* \in P^{[\alpha]}, \quad v_* \in Q \qquad (1.9)$$

We assume, further, the fulfillment of the following conditions.

1. For a specified initial position $\{t_0, w_0\} = \{t_0, z_0\}$ the second player's problem of performing the motions $w[t]$ to encounter M within the limits of N is solvable for the system (1.8), (1.9), i.e. a position strategy $V \div v(t, w)$ [2] exists which ensures the contact

$$\{w[\tau^*]\} \in M, \quad \{w[t]\} \in N \quad (t_0 \leqslant t \leqslant \tau^* \leqslant \vartheta) \qquad (1.10)$$

(When M is present we assume ϑ as a finite value; when M is absent we can have $\vartheta = \infty$.)

2. The system

$$s^{\cdot} = A s - B p(s) + C q(s) \qquad (1.11)$$

is stabilizable (see [5], p. 477), i.e. system (1.11) can be made asymptotically Liapunov-stable (see [6], p. 56) by a suitable choice of linear functions $p(s) = p_0' s$ and $q(s) =$

$q_0's$ (the prime denotes transposition).

3. The system

$$y^{\cdot} = D_2 y \tag{1.12}$$

is asymptotically Liapunov-stable.

In the general case, according to [2], when Condition 1 is fulfilled for any finite ϑ, for the system (1.8), (1.9) there exists a v-stable bridge W, lying in N on which the second player can hold all motions $w[t]$ up to contact with M inside N for $\tau^* \leqslant \vartheta$. If M is absent the existence of a suitable v-stable bridge W, lying in N, follows from [7] under Condition 1 (where now the first inclusion in (1.10) is not required) with $\vartheta < \infty$ and $\vartheta = \infty$.

According to Liapunov's theorem (see [6], p. 79), for the asymptotically stable system

$$s^{\cdot} = As - Bp_0's + Cq_0's \tag{1.13}$$

we can find, for any preselected negative-definite quadratic form $\beta(s)$, a positive-definite quadratic form $\lambda(s)$ for which the equality

$$\left(\frac{d\lambda}{dt}\right)_{(1.13)} = \left(\frac{\partial\lambda}{\partial s}\right)'(As - Bp_0's + Cq_0's) = \beta(s) \tag{1.14}$$

$$\left(\beta(s) = \sum_{i,j=1}^{k} \beta_{ij} s_i s_j, \quad \lambda(s) = \sum_{i,j=1}^{k} \lambda_{ij} s_i s_j\right)$$

is fulfilled. Here $(d\lambda / dt)_{(1.13)}$ is the total time derivative of function $\lambda(s)$ by virtue of system (1.13).

The following is the main result. When Conditions 1 – 3 are fulfilled, relying on the v-stable bridge W and on the function $\lambda(s)$, we can organize, for any preselected $\varepsilon > 0$, a leader-control procedure

$$V \div \{v(\tau, z, w), \quad u_*(\tau, z, w), \quad v_*(t, \tau, z, w, u_*(\cdot))\}$$

such that condition (1.5) is fulfilled for all motions $z_\Delta[t] = z_\Delta[t, t_0, z_0, V, u[\cdot]]$ generated by this procedure, provided that the parameter μ is sufficiently small and the t-axis partitioning step $\delta = \sup_i (\tau_{i+1} - \tau_i)$ $(i = 0, 1, \ldots)$ also is sufficiently small.

2. We construct the desired procedure V in the following manner. We set up one more auxiliary system

$$z^{*\cdot} = Az^* + Bu^* + c^* \tag{2.1}$$

where the controls u^* and c^* are subject to the conditions

$$u^* \in P, \qquad c^* \in \Gamma^* \tag{2.2}$$

As the set Γ^* in (2.2) we choose a strictly convex set containing the set $\Gamma = \{c : c = Cv,\ v \in C^{[\alpha]}\}$ and approximating it so that for any vector $c^* \in \Gamma^*$ the vector $c \in \Gamma$ closest to it satisfies the inequality $\| c - c^* \| \leqslant \zeta$, where $\zeta > 0$ is some sufficiently small number, the sense of smallness of which is clarified below. In addition, we require that a vector $c^{*\circ} \in \Gamma^*$, satisfying the condition

$$l'c^{*\circ} = \min_{c^* \in \Gamma^*} l'c^* \tag{2.3}$$

satisfy a Lipschitz condition with respect to l. The indicated choice of set Γ^* in Γ and

$\zeta > 0$ is always possible. We assume further (now, as a matter of fact, without loss of generality) that the mapping $c = Cv$ with $c \in \Gamma$ and $v \in Q^{[\alpha]}$ is one-to-one.

Let us compute, as yet completely formally, the total time derivative $d\lambda / dt$ of the function $\lambda(s)$, setting $s = z^* - w$ in it, where z^* and w are the solutions of Eqs. (2.1) and (1.8), respectively. We obtain

$$\frac{d\lambda}{dt} = \left(\frac{\partial\lambda}{\partial s}\right)'_{s=z^*-w} (A(z^* - w) + B(u^* - u_*) + c^* - Cv_*) \tag{2.4}$$

We choose $u_*^\circ(z^*, w)$ from the maximum condition

$$\max_{u_* \in P^{[\alpha]}} \left(\frac{\partial\lambda}{\partial s}\right)' Bu_* = \left(\frac{\partial\lambda}{\partial s}\right)' Bu_*^\circ \tag{2.5}$$

for any value of τ with $z^* = z^*[\tau]$ and $w = w[\tau]$. Next, from the value u_*° we choose, assuming (as yet formally) that the inclusion $\{\tau, w[\tau]\} \in W$ is fulfilled at instant τ, for some interval $\tau \leqslant t \leqslant \tau + \delta$ the function

$$v_*[t] = v_*^\circ(t, \tau, z^*, w, u_*^\circ)\ (z^* = z^*[\tau],\ \ w = w[\tau])$$

from the condition either of retention of the motion $w[t]$ of (1.8) on bridge W for $\tau \leqslant t \leqslant \tau + \delta$ or of contact with M for $\tau^* < \tau + \delta$. Such a choice of the control $v_* = v_*^\circ$ is possible as a consequence of the v-stability of bridge W (see [2]). We select the "control" $c_0^*(z^*, w)$ from the minimum condition

$$\min_{c^* \in \Gamma^*} \left(\frac{\partial\lambda}{\partial s}\right)' c^* = \left(\frac{\partial\lambda}{\partial s}\right)' c_0^* \tag{2.6}$$

From the chosen $c_0^*(z^*, w)$ we select the vector $c_0(z^*, w) \in \Gamma$ closest to it and next from $c_0(z^*, w)$ we determine $v^\circ \in Q^{[\alpha]}$ for $\| z^* - w \| \geqslant \nu_* > 0$ by complementing the construction of $v^\circ \in Q^{[\alpha]}$ for $\| z^* - w \| < \nu_*$ so that the function $v^\circ(z^*, w)$ satisfies the Lipschitz condition. The meaning of the sufficiently small constant $\nu_* > 0$ is clarified below. We note that by the given constructions the vector $v^\circ(z^*, w)$ will satisfy the Lipschitz condition with respect to $z^* - w$ and the inequality

$$\| c_0^* - Cv^\circ \| \leqslant \zeta \tag{2.7}$$

will also be satisfied.

The control of the system composed from object (1.4) and leader (1.8) is effected in a discrete scheme $\tau_i \leqslant t < \tau_{i+1}$ $(i = 0, 1, \ldots, \ \tau_0 = t_0)$ as follows. At the initial instant we assume $\{t_0, w_0\} = \{t_0, z_0\}$, then on each semi-interval $\tau_i \leqslant t < \tau_{i+1}$ the leader's phase vector $w_\Delta[t]$ is varied in accordance with Eq. (1.8), where

$$u_* = u_*^\circ(z_\Delta[z_i],\ \ w_\Delta[\tau_i]) \tag{2.8}$$

$$v_*[t] = v_*^\circ(t, \tau_i, z_\Delta[\tau_i], w_\Delta[\tau_i], u_*^\circ(z_\Delta[\tau_i], w_\Delta[\tau_i])) \tag{2.9}$$

The object's phase vector $\{z_\Delta[t],\ y_\Delta[t]\}$ is varied in accordance with Eq. (1.4), where

$$v = v^\circ(z_\Delta[\tau_i],\ \ w_\Delta[\tau_i]) \tag{2.10}$$

according to the problem's statement, the control $u = u[t]$ in (1.4) is developed by the first player on the basis of some control rule selected by him, and we can run into any measurable realization $u = u[t]$ constrained only by the condition $u[t] \in P$.

However, from the sense of the problem analyzed here, the controls u_* of (2.8), v_* of (2.9), and v of (2.10) are developed by the second player.

3. Let us show that under Conditions 1 – 3 described in Sect. 2 the procedure $V \div \{v^\circ, u_*{}^\circ, v_*{}^\circ\}$ for choosing the controls v, u_* and v_* for every preselected value $\varepsilon > 0$ ensures, with a suitable choice of $\zeta > 0$, $\nu_* > 0$, $\mu > 0$ and $\delta > 0$, the ε-proximity of motions $z_\Delta[t]$ and $w_\Delta[t]$ for $t \geqslant t_0$ and, starting from a certain instant $t = t_0 + \delta^*$ $(\delta^* > 0)$, ensures also the proximity of motions $y_\Delta[t]$ and $y_\Delta{}^\circ(z_\Delta[t], v)$; $\delta^* > 0$ is any preselected small quantity.

As a matter of fact, as a consequence of the asymptotic stability of system (1.12), by virtue of Liapunov's theorem (see [6], p. 79) we can find, for any preselected negative-definite quadratic form $\eta(y)$, a positive-definite quadratic form $\xi(y)$, whose total derivative $(d\xi / dt)_{(1.12)}$ by virtue of Eq. (1.12) satisfies the condition

$$\frac{d\xi}{dt} = \eta(y), \quad \eta(y) = \sum_{i,j=1}^{n-k} \eta_{ij} y_i y_j, \quad \xi(y) = \sum_{i,j=1}^{n-k} \xi_{ij} y_i y_j \tag{3.1}$$

We now set

$$s = z_\Delta[t] - w_\Delta[t], \quad y^* = y_\Delta[t] - y_\Delta{}^\circ(z_\Delta[t],\ v(z_\nabla[t],\ w_\Delta[t]))$$

and we set up the equations of perturbed motion on the semi-interval $[\tau_i, \tau_{i+1})$

$$ds_\Delta[t] / dt = As_\Delta[t] + D_1 y_\Delta{}^*[t] + B(u - u_*) + C(v - v_*) \tag{3.2}$$

$$\frac{dy_\Delta^*[t]}{dt} = \frac{1}{\mu} D_2 y_\Delta^*[t] - \frac{dy_\Delta^\circ(z_\Delta[t], v)}{dt}$$

All the differentiations in (3.2) are legitimate since under the choice of controls made, the corresponding functions are absolutely continuous for $\tau_i < t < \tau_{i+1}$. Therefore, equality (3.2) makes sense for almost all values from each semi-interval $\tau_i \leqslant t < \tau_{i+1}$. Equality (3.2) can be preserved when passing to the whole semiaxis $t \geqslant t_0$, however we need take into account that at the instants $t = \tau_i$, as a consequence of the stepwise variations

$$v[\tau_i] - v[\tau_i - 0] = v(z[\tau_i],\ w[\tau_i]) - v(z[\tau_{i-1}],\ w[\tau_{i-1}])$$

it is necessary to treat now the term $dy^\circ(z, v) / dt$ as a generalized derivative which contains a term of the form $\varkappa_i \delta(t - \tau_i)$, where $\delta(t - \tau_i)$ is the impulse δ-function. Here the vector $\varkappa_i$ satisfies the estimate

$$\|\varkappa_i\| \leqslant K_0 \delta, \quad K_0 = \text{const}, \quad \delta = \sup_i(\tau_{i+1} - \tau_i) \quad (i = 0, 1, \ldots) \tag{3.3}$$

We estimate the change in the Liapunov function

$$\gamma(s, y^*) = \lambda(s) + \xi(y^*) \tag{3.4}$$

by virtue of (3.2), where the controls u_*, v_*, v are selected in accordance with formulas (2.8) – (2.10). By computing, now no longer formally, the total time derivative $(d\gamma / dt)_{(3.2)}$ of the function $\gamma(s, y^*)$ (3.4) by virtue of system (3.2), we obtain

$$\left(\frac{d\gamma}{dt}\right)_{(3.2)} = \left(\frac{\partial \lambda}{\partial s}\right)' (As_\Delta[t] + D_1 y_\Delta^*[t] + B(u[t] - u_*(s_\Delta[\tau_i])) + C(v(s_\Delta[\tau_i]) - v_*[t])) + \left(\frac{\partial \xi}{\partial y^*}\right)' \left(\frac{1}{\mu} D_2 y_\Delta^*[t] - \frac{dy_\Delta^\circ(z_\Delta[t], v(s_\Delta[\tau_i]))}{dt}\right) \quad (\tau_i \leqslant t < \tau_{i+1}) \tag{3.5}$$

Here at the instants $t = \tau_i$ the component $\xi(y^*[t])$ in the function $\gamma(s[t], y^*[t])$ further undergoes, as a consequence of (3.3), the jumps

$$\xi(y^*[\tau_i]) - \xi(y^*[\tau_i - 0]) \leqslant K_0\delta \tag{3.6}$$

Taking (1.14), (2.5) – (2.7) and (3.1) into account, we can verify that the estimates

$$\begin{aligned}\frac{d\gamma}{dt} \leqslant \beta(s_\Delta[t]) + \frac{1}{\mu}\eta(y_\Delta^*[t]) + K_1\|s_\Delta[t]\|\,\|y_\Delta^*[t]\| + {}&\\ K_2\|y_\Delta^*[t]\| + K_3\zeta\|s_\Delta[t]\| + K_4(1 + \|s_\Delta[t]\| + \|y_\Delta^*[t]\|)\delta + {}&\\ K_5(\tau_i - \tau_{i-1})^2 \qquad \text{for } \nu_* \leqslant s_\Delta[t]\| \leqslant \nu^*&\end{aligned} \tag{3.7}$$

$$\begin{aligned}\frac{d\gamma}{dt} \leqslant \frac{1}{\mu}\eta(y_\Delta^*[t]) + L_1(1 + \|y_\Delta^*[t]\|)\delta + {}&\\ L_2\|y_\Delta^*[t]\| + L_3\nu_*^2 + L_4\zeta\nu_* \quad \text{for } \|s_\Delta[t]\| < \nu_*&\end{aligned} \tag{3.8}$$

are valid on the semi-interval $\tau_i \leqslant t < \tau_{i+1}$. Here K_j $(j = 1, \ldots, 5)$ and L_j $(j = 1, \ldots, 4)$ are constants, ν^* is so small a constant that the inequality

$$\begin{aligned}\min_{v\in Q^{[\alpha]}}\left(\frac{\partial\lambda}{\partial s}\right)' Cv + \max_{u\in P}\left(\frac{\partial\lambda}{\partial s}\right)' Bu - \min_{v\in Q}\left(\frac{\partial\lambda}{\partial s}\right)' Cv - {}&\\ \max_{u\in P^{[\alpha]}}\left(\frac{\partial\lambda}{\partial s}\right)' Bu < \left(\frac{\partial\lambda}{\partial s}\right)'[-Bp_0's + Cq_0's]&\end{aligned} \tag{3.9}$$

is fulfilled in the region $\|s_\Delta[t]\| \leqslant \nu^*$

With due regard to inequality (3.6) and by arguments which are standard in the theory of stability of motion [5, 6, 8] and in the theory of differential equations with a small parameter [1], it is not difficult to show the validity of the following statement. For any arbitrarily small $\varepsilon > 0$ we can find so small values of $\nu_*(\varepsilon) > 0$, $\zeta(\varepsilon) > 0$ and values of $\mu_0(\varepsilon) > 0$, $\delta_0(\varepsilon, \mu) > 0$ such that for $\mu \leqslant \mu_0(\varepsilon)$ the procedure described for choosing the controls, developed in the leader-control scheme, ensures the fulfillment of the inequalities

$$\|s_\Delta[t]\| \leqslant \varepsilon \quad \text{for} \quad t \geqslant t_0; \qquad \|y_\Delta^*[t]\| \leqslant \varepsilon \quad \text{for} \quad t \geqslant t_0 + \delta \tag{3.10}$$

for all motions $s_\Delta[t]$ and $y_\Delta^*[t]$, generated by this procedure and having the partitioning step $\sup_i(\tau_{i+1} - \tau_i) \leqslant \delta_0(\varepsilon, \mu)$. Since here the choice of control $v_*(t, \tau_i, z_\Delta[\tau_i], w_\Delta[\tau_i], u_*(\cdot))$ ensures the retention of motion $w_\Delta[t]$ on W, we obtain the following result.

Theorem. Let Conditions 1 – 3 be fulfilled. Then for every $\varepsilon > 0$ we can organize a control procedure

$$V \div \{v(\tau, z, w), \quad u_*(\tau, z, w), \quad v_*(t, \tau, z, w, u_*(\cdot))\} \tag{3.11}$$

which for sufficiently small $\mu(\varepsilon) > 0$ and $\delta(\varepsilon, \mu) > 0$ ensures the fulfillment of the conditions

$$\{z_\Delta[\tau^*]\} \in M^{[\varepsilon]}, \quad \{z_\Delta[t]\} \in N^{[\varepsilon]} \qquad (t_0 \leqslant t \leqslant \tau^*)$$

for all motions $z_\Delta[t] = z_\Delta[t, t_0, z_0, V, u[\cdot]]$, generated by this procedure.

4. In conclusion we examine an example illustrating the method described for constructing a control with leader for a system described by differential equations containing a small parameter in a part of the derivatives.

Suppose that there are two material points $m^{(1)}$ and $m^{(2)}$ moving on some plane Π. The material point $m^{(1)}$ of unit mass pursues the material point $m^{(2)}$ of mass μ experiencing the medium's drag which is linear in rate with a unit proportionality coefficient. On each of the points acts its own control force $F^{(1)} = u$ and $F^{(2)} = v$, respectively; these forces are subject to the constraints $\| u \| \leqslant h^{(1)}$, $\| v \| \leqslant h^{(2)}$.

Let $\rho^{(i)}$ $(i = 1, 2)$ be the radius-vectors of points $m^{(i)}$. By introducing the notation

$$\rho^{(1)} - \rho^{(2)} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \quad \rho^{\cdot(1)} = \begin{bmatrix} z_3 \\ z_4 \end{bmatrix}, \quad \rho^{\cdot(2)} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

we obtain the equations of motion of the point $m^{(1)}$ and $m^{(2)}$ in the form of the following system of differential equations:

$$z_1^{\cdot} = z_3 - y_1, \quad z_2^{\cdot} = z_4 - y_2, \quad z_3^{\cdot} = u_1, \quad z_4^{\cdot} = u_2 \qquad (4.1)$$
$$\mu y_1^{\cdot} = -y_1 + v_1, \quad \mu y_2^{\cdot} = -y_2 + v_2$$

where $\mu > 0$ is a small parameter. We look at the evasion problem for the second player striving to prevent contact of the material points $m^{(1)}$ and $m^{(2)}$ along the geometric coordinates during the interval $t_0 \leqslant t \leqslant \vartheta$, where ϑ is an arbitrarily large fixed number. Conditions (1.5) here have the form

$$\{z_\Delta [t]\} \in N^{[\varepsilon]}, \qquad t_0 \leqslant t \leqslant \vartheta \qquad (4.2)$$

where the set N is determined by the inequality $z_1^2 + z_2^2 \geqslant \sigma^2, \sigma > 0$ is a sufficiently small number and $\varepsilon < \sigma$.

Let us pass on the constructing the desired procedure V for choosing the controls ensuring the fulfillment of condition (4.2). Setting $\mu = 0$ in (4.1), we determine the vector-valued function $y^\circ (z, v)$ of (1.6), having the form $y^\circ (z, v) = v$. We set up the auxiliary system of form (1.7), described here by the equations

$$z_1^{\cdot\circ} = z_3^\circ - v_1, \quad z_2^{\cdot\circ} = z_4^\circ - v_2, \quad z_3^{\cdot\circ} = u_1, \quad z_4^{\cdot\circ} = u_2 \qquad (4.3)$$

We compare Eqs. (4.3) with the equations of motion of the leader of form (1.8). We obtain

$$w_1^{\cdot} = w_3 - v_{1*}, \quad w_2^{\cdot} = w_4 - v_{2*}, \quad w_3^{\cdot} = u_{1*}, \quad w_4^{\cdot} = u_{2*} \qquad (4.4)$$

where the controls u_* and v_* are constrained by the conditions

$$\| u_* \| \leqslant h^{(1)} + \alpha, \quad \| v_* \| \leqslant h^{(2)} - \alpha \quad (\alpha > 0) \qquad (4.5)$$

Suppose that we are given some initial position $\{t_0, z_0\}$. We compare it with the initial position $\{t_0, w_0\} = \{t_0, z_0\}$. According to [4] the second player can always choose in system (4.4) his own control v_* for the chosen position $\{t_0, w_0\}$ in such a way that for any choice, of which he is informed, of the first player's control u_*, he ensures the evasion

$$w_1^2 [t] + w_2^2 [t] > \sigma^2 \qquad (4.6)$$

where $\sigma > 0$ is a sufficiently small number depending on the initial data and on the fixed ϑ. In particular, for example [9], at each instant $t \in [t_0, \vartheta]$ he can choose the control $v_* [t]$ from the following conditions:

$$(v_*, w^{(1)}) = 0, \quad (v_*, w^{(2)}) \leqslant 0, \quad \| v_* \| = h^{(2)} - \alpha \qquad (4.7)$$
$$w^{(1)} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \quad w^{(2)} = \begin{bmatrix} w_3 \\ w_4 \end{bmatrix}$$

where $(v_*, w^{(i)})$ is the scalar product of vectors v_* and $w^{(i)}$. Then, according to the results in [7] v-stable bridge W exists for system (4.4), which passes through the initial position $\{t_0, w_0\}$ and lies in N for $t_0 \leqslant t \leqslant \vartheta$, i.e. Condition 1 is fulfilled. Here it is sufficient to choose control v exactly from conditions (4.7) to retain the motions $w[t]$ on bridge W.

To stabilize a system of form (1.11) described here by the equations

$$s_1^{\cdot} = s_3 - q_1(s), \quad s_2^{\cdot} = s_4 - q_2(s), \quad s_3^{\cdot} = -p_1(s), \quad s_4^{\cdot} = -p_2(s)$$

it is sufficient to set

$$q_1(s) = s_1 + s_3, \quad q_2(s) = s_2 + s_4, \quad p_1(s) = s_3, \quad p_2(s) = s_4$$

i.e. here Condition 2 is fulfilled as well. Having specified the negative-definite quadratic form $\beta(s)$ (see 1.14) in the form, for example,

$$\beta(s) = -\sum_{i=1}^{4} s_i^2$$

we find the positive-definite quadratic form $\lambda(s)$ which in this case has the form

$$\lambda(s) = \frac{1}{2}\sum_{i=1}^{4} s_i^2$$

The system of form (1.12), described here by the equations

$$y_1^{\cdot} = -y_1, \quad y_2^{\cdot} = -y_2$$

is asymptotically Liapunov-stable, which signifies the fulfillment of Condition 3.

Thus, Conditions 1 – 3 of the theorem are fulfilled for the given problem; consequently, the second player can so organize the control procedure (3.15) with leader that it ensures the fulfillment of condition (4.2) for sufficiently small values of the parameter $\mu(\varepsilon)$ and for a sufficiently small partitioning step $\delta(\varepsilon, \mu)$.

This procedure is that the second player selects the control $v_*[t]$ from conditions (4.7), while the controls u_* and v, by the following formulas:

$$u_{i*} = \frac{(h^{(1)} + \alpha)\, s_{2+i}}{\sqrt{s_3^2 + s_4^2}}, \quad v_i = \frac{h^{(2)} s_i}{\sqrt{s_1^2 + s_2^2 + \zeta_0 (s_3^2 + s_4^2)}} \qquad (i = 1, 2)$$

for $0 < \nu_* \leqslant \|s\| \leqslant \nu^*$, where ζ_0 is a sufficiently small number. As regards the magnitudes of the small constants ν_*, ν^* and ξ_0, to find them it is necessary to write out expression (3.5) explicitly for the given example, to estimate it, and to obtain the desired values from this estimate.

We note further that since it is possible to ensure evasion (4.6) in the system described by Eqs. (4.4) during the infinite interval $t_0 \leqslant t < \infty$ [10] for a sufficiently small $\sigma > 0$, the described procedure for constructing the controls v, u_* and v_* ensures evasion (4.2) for all $t \geqslant t_0$ in the system described by Eqs. (4.1).

REFERENCES

1. Tikhonov, A.N., Systems of differential equations containing small parameters in the derivatives. Matem.Sb., Vol.31, №3, 1952.
2. Krasovskii, N.N., Differential games of encounter-evasion. I, II. Izv.Akad. Nauk SSSR, Tekhn.Kibernetika, №№2, 3, 1973.
3. Krasovskii, N.N. and Subbotin, A.I., Approximation in a differential game. PMM Vol.37, №2, 1973.

4. Pontriagin, L.S. and Mishchenko, E.F., The problem of one controlled object escaping from another. Dokl. Akad. Nauk SSSR, Vol. 189, №4, 1969.
5. Malkin, I.G., Theory of Stability of Motion. Moscow, "Nauka", 1966.
6. Liapunov, A.M., General Problem of the Stability of Motion. Moscow-Leningrad, Gostekhizdat, 1950.
7. Krasovskii, N.N., On an evasion game problem. Differentsial'nye Uravneniia, Vol. 8, №2, 1972.
8. Barbashin, E.A., On stability with respect to impulse actions. Differentsial'nye Uravneniia, Vol. 2, №7, 1966.
9. Barabanova, N.N. and Subbotin, A.I., On continuous evasion strategies in game problems on the encounter of motions. PMM Vol. 34, №5, 1970.
10. Gusiatnikov, P.B., On the l-evasion of contact in a linear differential game. PMM Vol. 38, №3, 1974.

Translated by N.H.C.

UDC 62-50

LINEAR PURSUIT PROBLEM UNDER LOCAL CONVEXITY CONDITIONS. SOLUTION OF THE SYNTHESIS EQUATION

PMM Vol. 38, №5, 1974, pp. 780-787
P.B. GUSIATNIKOV
(Moscow)
(Received March 1, 1973)

We derive conditions sufficient for the completion of pursuit by the time-independent feedback principle; the paper relates closely to the investigations in [1 – 7].

1. Let a linear pursuit problem in the n-dimensional Euclidean space R be described a) by the linear vector differential equation

$$dz / dt = Cz - u + v \tag{1.1}$$

where C is an nth-order constant square matrix; $u = u(t) \in P$ and $v = v(t) \in Q$ are vector-valued functions, measurable for $t \geqslant 0$, called the controls of the players (the pursuer and the pursued, respectively); $P \subset R$ and $Q \subset R$ are convex compacta;

b) by a terminal set M representable in the form $M = M_0 + W_0$, where M_0 is a linear subspace of space R, W_0 is some compact convex set in a subspace L which is the orthogonal complement to M_0 in R. By π we denote the operator of orthogonal projection onto L; we denote the dimension of L by ν and the unit sphere in L by K. We assume that $\nu \geqslant 2$. We denote the matrix e^{tC} by $\Phi(t)$. Every Carathéodory-solution $z(t)$ [1], $T_1 \leqslant t \leqslant T_2$, of Eq. (1.1) with the initial condition $z(T_1) = z_0$ is called a motion and denoted $z(t) = z(t;\ T_1, z_0, u, v, T_2)$.

The pursuer's aim is to bring point z onto set M; the pursued tries to prevent this. We say that the pursuit from a point z_0 can be concluded in a time $t(z_0)$ if there exists a vector-valued function $u(z) \in P$ (called the "synthesis"), defined on the whole space R, such that for arbitrary pursued's control $v(t)$, the pursuer by applying the control